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AN ADDENDUM TO THE PAPER: DYNAMIC GROWTH OF AN EDGE CRACK IN A HALF SPACE

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1. INTRODUCTION

The plane strain problem of an edge crack, starting at the surface of a half-space and propagating with a constant speed, was considered in [1]. The method described was based on a superposition of dislocations (of constant Burgers vector), each propagating at a constant velocity perpendicular to the surface of the half space (see Fig. 1 of [1]). All dislocations appeared at the surface at time $t = 0$.

For the method described, displacements are homogeneous functions of degree zero in time t and the spatial coordinates x and z; that is, the displacements, along $x = 0$ say, are of the form $f(t/z)$. This implies that stresses are of the form $G(t/z)/z$ along $x = 0$ and $F(t/x)/x$ along $z = 0$. Since the problems of greatest interest (as presented in loading cases 1–4 in [1]) are problems in which the stresses are homogeneous functions of degree zero in t, x and z, a method was proposed for modifying the above dislocation solution in an attempt to take these problems of homogeneous stress distributions into account.

In retrospect, it is obvious that the method proposed in [1] is not correct for stress fields which are homogeneous of degree zero and that the derivation in [1] only applies to problems in which displacements are homogeneous functions of degree zero in t, x and z.

Two loading cases which are applicable to the derivation in Sections 1–6 of [1] are given below. A generalization of this method is then given for the problems in which stresses are homogeneous functions of degree zero in t, x and z.

2. LOADING CASES APPLICABLE TO METHOD IN [1]

Case 1

Two normal point loads propagate out from the origin in opposite directions on the surface of the half space at a constant velocity u. The boundary conditions on the half-space are

$$\sigma_{zz}(x, 0, t) = \begin{cases} \Delta\mu\,\delta(ut - x), & x > 0, \\ \Delta\mu\,\delta(ut + x), & x < 0, \end{cases}$$
$$\sigma_{xz}(x, 0, t) = 0, \quad -\infty < x < \infty. \tag{2.1}$$

Zero initial conditions are assumed in all problems that follow. The value of $\sigma_{xx}(0, z, t)$ on the symmetry line (which is also the line along which the crack propagates) normal to the surface is

$$\sigma_{xx}(0, z, t) = \frac{2\Delta\mu}{\pi z u^2}\left\{\frac{(b^2 - 2w^2 + 2w^2)(b^2 - 2w^2)w^2}{(d^2 - a^2 + w^2)R[i(w^2 - a^2)^{1/2}](w^2 - a^2)^{1/2}} H(w - a) + \frac{4(w^2 + b^2)^{1/2}(a^2 - b^2 + w^2)^{1/2} w^2}{(d^2 - b^2 + w^2)\,R[i(w^2 - b^2)^{1/2}]} H(w - b)\right\}_{w = t/z}, \tag{2.2}$$

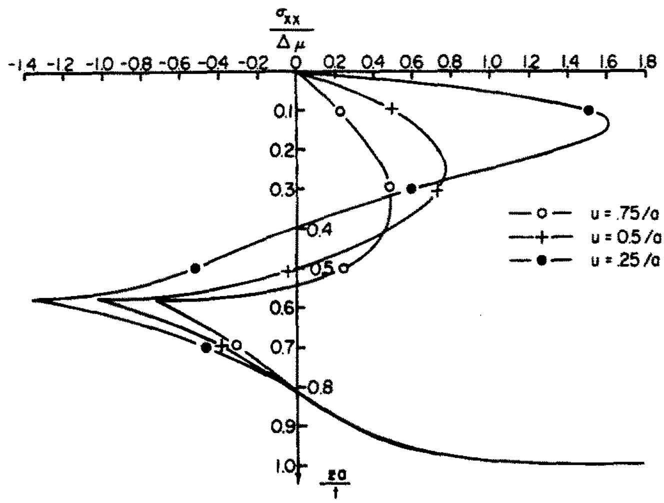

Fig. 1.

where $d = 1/u$ and a, b are the longitudinal and shear wave slownesses, respectively. R is the Rayleigh function as defined in [1]. Plots of $\sigma_{xx}(0, z, t)$ are given in Fig. 1 for three different values of u. In all the calculations $b^2 = 3a^2$ was used, corresponding to a Poisson's ratio of 0.25.

Case 2

This is the shear problem corresponding to Case 1. The boundary conditions on the half-space are

$$\sigma_{xz}(x, 0, t) = \begin{cases} -\Delta\mu\,\delta(ut - x), & x > 0, \\ \Delta\mu\,\delta(ut + x), & x < 0, \end{cases}$$

$$\sigma_{zz}(x, 0, t) = 0, \quad -\infty < x < \infty. \tag{2.3}$$

The value of $\sigma_{xx}(0, z, t)$ in the half-space is

$$\sigma_{xx}(0, z, t) = \frac{4\Delta\mu}{\pi z u}\left\{\frac{(w^2 - a^2)^{1/2}(b^2 - a^2 + w^2)^{1/2} w}{(d^2 + a^2 + w^2)R[i(w^2 - a^2)^{1/2}]} H(w - a) - \frac{(w^2 - b^2)^{1/2}(b^2 - 2w^2)w^2}{(d^2 - b^2 + v^2)R[i(w^2 - b^2)^{1/2}]}\right\}_{w = t/z}, \tag{2.4}$$

and $d = 1/u$. These values are plotted versus za/t in Fig. 2 and the stress intensity factors versus $V_{CT}b$ for cases 1 and 2 with speeds of load point propagation $u = 0.25/a$, $0.5/a$, $0.75/a$ are shown in Fig. 3. V_{CT} is the speed of propagation of the crack.

3. FUNDAMENTAL SOLUTION FOR CASE WHEN STRESSES ARE HOMOGENEOUS FUNCTIONS OF DEGREE ZERO

When the applied tractions are homogeneous functions of degree zero in t, x and z, the stresses corresponding to the fundamental solution must also be of this type. A suitable type of dislocation on which to base a superposition scheme is that used by Freund [2], called a velocity dislocation (to indicate that it is velocity that is discontinuous). However, an alternative choice (and one simpler to apply numerically, although no more general) is to use a dislocation with Burgers vector (displacement discontinuity) proportional to time.

Proceeding as in [1], the following problem is solved in a full-space, with coordinate axes aligned with those in the half-space. The boundary conditions are

$$u_x(0, z, t) = \begin{cases} \Delta t\, H(vt - z), & z > 0, \\ \Delta t\, H(vt + z), & z < 0, \end{cases}$$

$$\sigma_{xz}(0, z, t) = 0, \quad -\infty < z < \infty. \tag{3.1}$$

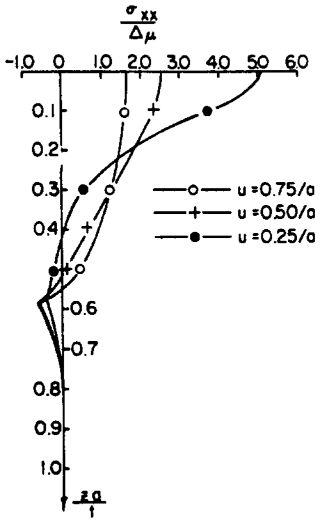

Fig. 2.

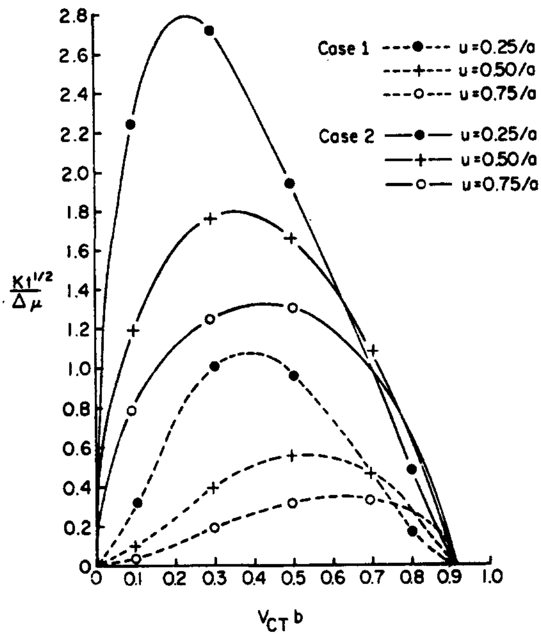

Fig. 3.

Solving as in [1] or by means of any other method, such as that in [3], the relevant stress components are:

$$\sigma_{xx}^{D}(x, z, t; v) = \frac{-\Delta\mu}{\pi b^2} 4d^2 \int_0^t \left\{ \frac{(b^2 - 2\lambda^2)^2}{\alpha} \frac{1}{(d^2 - \lambda^2)^2} \frac{\partial\lambda}{\partial\tau} H(\tau - ra) + \frac{4\,\eta^2 \beta}{(d^2 - \eta^2)^2} \frac{\partial\eta}{\partial\tau} H(\tau - rb) \right\} \mathrm{d}\tau, \tag{3.2}$$

where

$$\lambda = -z\tau/r^2 + i\,x/r\,(\tau^2/r^2 - a^2)^{1/2},$$

$$r = (x^2 + z^2)^{1/2},\ \alpha = (a^2 - \lambda^2)^{1/2},\ \beta = (b^2 - \eta^2)^{1/2},$$

$d = 1/v$ and

$$\sigma^D_{zz}(x, 0, t; v) = \frac{-\Delta\mu}{\pi b^2} 4d^3 \int_0^{t/x} \left\{ \frac{(b^2 - 2\omega^2)(b^2 - 2a^2 + 2\omega^2)}{(d^2 - a^2 + \omega^2)(\omega^2 - a^2)^{1/2}} H(\omega - a) + \frac{4(\omega^2 - b^2)^{1/2}\omega^2}{(d^2 - b^2 + \omega^2)^2} H(\omega - b) \right\} d\omega. \quad (3.3)$$

The function η is the same as λ with "a" replaced by "b".

The value of σ_{xx} when $x = 0$ is required and some care must be taken in obtaining it since there is a double pole at $\lambda = d$. However, if the integral is first evaluated with $x \neq 0$ and then $x \to 0$, it can be seen that there is a single pole at $t/z = d$, as expected. When the integral was evaluated numerically, it was split into two parts: one part was regular and could be evaluated numerically and the other part was singular, but could be evaluated analytically.

To cancel the stresses on the surface of the half-space, due to the above problem, the solution to the following problem is required. The boundary conditions on the half space are

$$\sigma_{zz}(x, 0, t) = \begin{cases} t\,\delta(ut - x), & x > 0 \\ t\,\delta(ut + x), & x < 0 \end{cases}$$

$$\sigma_{zx}(x, 0, t) = 0, \quad -\infty < x < \infty. \quad (3.4)$$

The relevant stress component is

$$\sigma^{PL}_{xx}(0, x, t; u) = \frac{2d^2}{\pi} \int_0^{t/z} \left\{ \frac{(d^2 + a^2 - \omega^2)(b^2 - 2a^2 + 2\omega^2)(b^2 - 2\omega^2)\omega}{(d^2 - a^2 + \omega^2)^2 R[i(\omega^2 - a^2)^{1/2}](\omega^2 + a^2)^{1/2}} H(\omega - a) + 4\frac{(d^2 + b^2 - \omega^2)(\omega^2 - b^2)^{1/2}(a^2 - b^2 + \omega^2)^{1/2}\omega}{(d^2 - b^2 + \omega^2)R[i(\omega^2 - b^2)^{1/2}]} H(\omega - b) \right\} d\omega \quad (3.5)$$

where $d = 1/u$.

The fundmental solution for a dislocation growing with time, propagating at a constant velocity v perpendicular to the half-space, having started at the half-space surface at time $t = 0$ can be written as in [1]; that is

$$\sigma^{FS}_{xx}(t/z; v) = \sigma^D_{xx}(t/z; v) - \int_0^{1/a} \sigma^D_{zz}(t/x = 1/u; v)\, \sigma^{PL}_{xx}(t/z; iu) du. \quad (3.6)$$

Superscript FS will stand for the fundamental solution and as indicated in eq. (3.5) the stress is a function of v and t/z only.

4. SINGULAR INTEGRAL EQUATION AND STRESSS INTENSITY FACTOR

Following [1], a singular integral equation can be formed;

$$-\sigma^{\text{Loads}}_{xx}(t/z) = \int_0^{V_{CT}} \sigma^{FS}_{xx}(t/z; v)\mu(v) dv, \; 0 \le z \le V_{CT}t, \quad (4.1)$$

where $\mu(v)$ is recognized as the slope of the crack face for fixed t/z. The same square root singular behaviour of $\mu(v)$ as $v \to 0$ and $v \to V_{CT}$ is assumed as in [1], so that $\mu(v) = F(v)/((V_{CT} - v)^{1/2} v^{1/2})$.

The stress intensity factor is given by

$$K = (2\pi)^{1/2} \frac{\Delta\mu}{b^2} (V_{CT}t)^{1/2} F(V_{CT}) \frac{\text{Re}[R(-1/V_{CT})]}{(1/V_{CT}^2 - a^2)^{1/2}} \quad (4.2)$$

5. RESULTS FOR SEVERAL LOADING CASES

The same loading cases will be considered as in [1]. Case 3 will be uniform pressure on the crack faces; case 4 will be a linearly varying pressure distribution on the crack faces, being zero at the surface and $\Delta\mu$ at the crack-tip; cases 5 and 6 will be uniform pressure and shear stress distributions on the half-space surface, spreading out with velocity u. Cases 3–6 correspond to cases 1–4 in [1] where more details are given. Plots of $\sigma_{xx}(0, z, t)$ for cases 5 and 6 are given in Figs. 3 and 4 of [1]. The equations for $\sigma_{xx}(0, z, t)$ for loading cases 5 and 6 are given below (to correct the typographical errors in the corresponding equations in [1]). For loading case 5:

$$\sigma_{xx}(0, z, t) = \frac{2\mu}{\pi u}\int_0^{t/z}\left\{\frac{(b^2-2a^2+2w^2)(b^2-2w^2)H(w-a)}{(d^2-a^2+w^2)R[i(w^2-a^2)^{1/2}](w^2-a^2)^{1/2}} + \frac{4(w^2-b^2)^{1/2}(a^2-b^2+w^2)w^2H(w-b)}{(d^2-b^2+w^2)R[i(w^2-b^2)^{1/2}]}\right\}\mathrm{d}w \quad (5.1)$$

and for loading case 6:

$$\sigma_{xx}(0, z, t) = \frac{4\mu}{\pi}\int_0^{t/z}\left\{\frac{(w^2-a^2)^{1/2}(b^2-a^2+w^2)^{1/2}(b^2-2w^2)w\,H(w-a)}{(d^2-a^2)R[i(w^2-a^2)^{1/2}]} - \frac{(w^2-b^2)^{1/2}(b^2-2w^2)w^2H(w-b)}{(d^2-b^2+w^2)R[i(w^2-b^2)^{1/2}]}\right\}\mathrm{d}w. \quad (5.2)$$

The numerical methods used were the same as in [1] except for the numerical integration where a method presented (with computer programs) in [5] was used. The ranges of integration where split into three parts; 0 to $0.1/b$, $0.1/b$ to $1/b$ and b to a and in each part 30 points were used in the program given in [5].

For the singular integral equation, 10 points were used along the crack. A few cases were checked using 20 points and the same accuracy of numerical integration and the difference in the stress intensity factors was less than 1.5%.

The results for the point loads represented in cases 1 and 2 show that for loads of the same nominal magnitude ($\Delta\mu$) and propagation speed, the stress intensity factor is greater for the shear point load than for the normal point load. See Fig. 3.

Comparison between the stress intensity factors for the linearly varying pressure distribution (case 4) on the crack faces and the constant pressure (case 3) can be made from Fig. 4 and shows that for a total applied force for case 4 of half the total applied force in case 3, the ratio of stress intensity factors is approximately 2/3 for medium crack velocities. This may have application in hydraulic fracture.

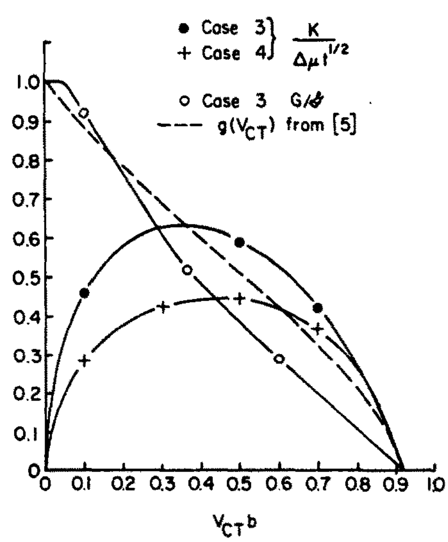

Fig. 4.

The ratio of the energy release rate as defined in [2], for case 4 and the static equivalent

$$\mathcal{G} = \frac{1-\nu^2}{E}\pi(1.12)^2 \Delta\mu\, l \tag{5.3}$$

where l is the instantaneous crack length, i.e. $V_{CT}t$, is shown in Fig. 4. The dynamic effect is more important than was erroneously concluded in [1], with the range of crack tip velocities for which a quasi-static approximation would give an error of less than 10% being reduced to $0 \le V_{CT} \le 0.1b$. As a comparison the value of $g(V_{CT})$ from [4] is also plotted, where $g(V_{CT})$ is the analogous quantity to $G/\mathcal{G}$ for a semi-infinite straight crack propagating in an infinite body.

The stress intensity factors for loading cases 5 and 6 are shown in Figs. 5 and 6. If the case of a normal compressive traction distribution on the half-space surface (as represented by case 5) is considered with frictional effects, such as might occur in the impact of a very soft object (e.g. a water droplet) on an elastic body and if a coefficient of friction of 0.3 is used as a typical value, it can be seen that the positive stress intensity factor due to the frictional (shear as in case 6) forces will be greater than the negative stress intensity factor due to the normal compressive load for a small range of crack tip velocities from 0 to approximately $0.1/b$. The conclusions are the same as in [1], i.e. if cracks initiate under impact loading as described, it will probably be due to effects such as friction and the initiation velocity will be low—probably in

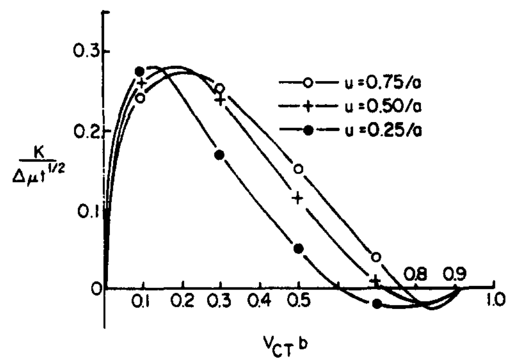

Fig. 5.

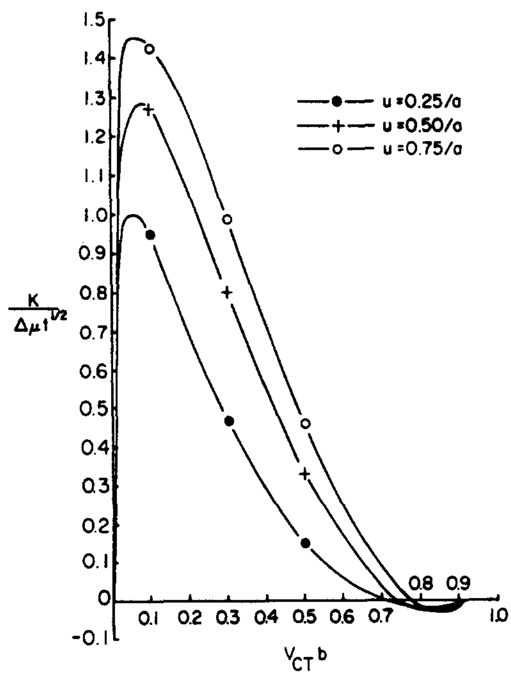

Fig. 6.

the range where quasi-static calculations will be sufficiently accurate for practical engineering purposes. This conclusion only applies to cracks as described.

The stress intensity factors in Figs. 5 and 6 are negative for crack tip velocities close to the Rayleigh wave speed. This does not appear to be a numerical error since doubling the number of points in the singular intergal equation only changed the results by less than 1.5%. The probable cause of the negative stress intensity factors can be seen by observing in Figs. 3 and 4 of [1] that at the position of the crack tip for these velocities, the stresses due to these loading cases are compressive.

A check was made on Koiter's result for the static problem of an edge crack in a half space by letting $V_{CT} \to 0$ as $t \to \infty$, so that $V_{CT}t \to l$, where l is crack length. The same check was made in [1] and although it appeared as if the correct limit was obtained for $V_{CT} \geq 0.05/b$ there, when $V_{CT} < 0.05/b$ the answers diverged. It was thought that a reason for this was found, but now it seems that this was not the most important cause of the problem. In the checks made with the revised calculations no problem was found in going to the static limit.

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